Analysis I Lecture 6

Evening sessions.

Séances du soir, MATH-101 analyse I pour sections d'ingénierie Automne 2025-2026. Semaines 4 à 13 comprises			
Jour	Horraire	Salle	Cours représentés
Lundi	17h30-19h00	CM 1 105	SV, SIE/GC/SC, MT, IN Strütt, Mila, Mounford, Lachowsa F, A, D, E
Mardi	17h30-19h00	BS 170	Inversée, EL/MX/CGC, IN, EN pi, G, E, en Friedli, Basterrechea, Lachowska, Monin
Mercredi	17h30-19h00	CO 122	GM, MT C, D Friedli, Mountford
Jeudi	18h15-19h45	MA B1 11	SIE/GC/SC, GM, EL/MX/CGC A, C, G Mila, Friedli, Basterrechea

Principe : Vous êtes bienvenu·es à n'importe quelle séance. Les cours représentés sont donnés à titre d'information si vous avez besoin de parler à un·e assistant·e de votre cours pour une question spécifique.

Complex numbers:

$$z = \underbrace{x}_{Re(z)} + i \cdot \underbrace{y}_{Im(z)} \qquad x, y \in \mathbb{R}$$

$$i = \sqrt{-1}$$

$$(x_1 + iy_1) + (x_2 + iy_2) = \underbrace{(x_1 + x_2)}_{Re z_1 + Re(z_2)} + i\underbrace{(y_1 + y_2)}_{Im z_1 + Im z_2}$$

$$(x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2)$$

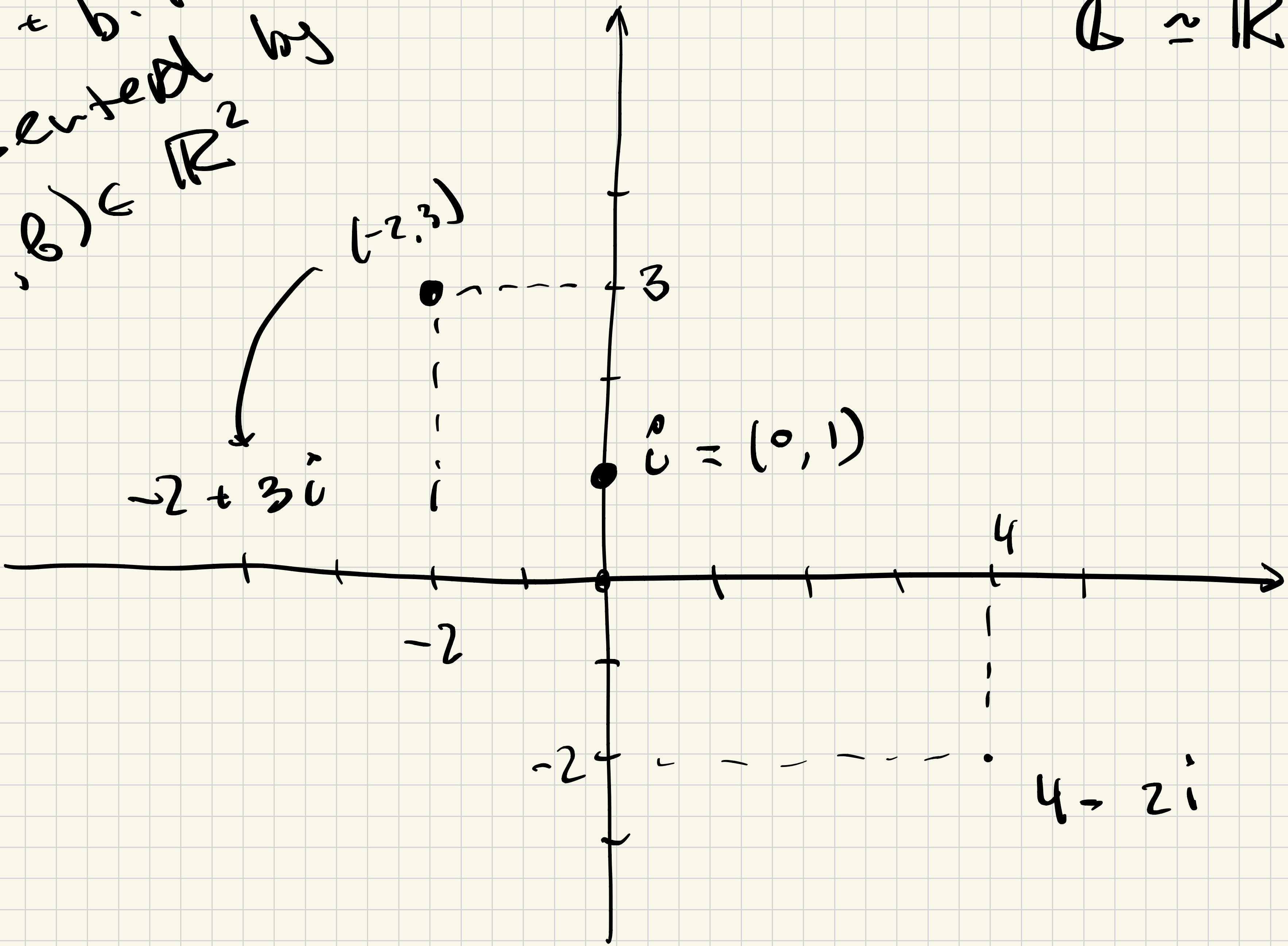
$$i^2 = -1$$

$z = a + b \cdot i$

represented by

$(a, b) \in \mathbb{R}^2$

$\mathbb{C} \cong \mathbb{R}^2$



Complex conjugation

Recall

Let $z = x + iy$ be a complex number

then $\mathrm{Re}(z) = x$

$\mathrm{Im}(z) = y$

The complex conjugate

$$\overline{z} = \overline{(x + iy)} = x - i \cdot y$$

Clearing denominators

Goal: Rewrite $\dfrac{z_1}{z_2}$ such that

denominator is purely real.

Recall $\dfrac{1}{z} = \dfrac{x - iy}{x^2 + y^2} = \dfrac{x}{x^2+y^2} - i\,\dfrac{y}{x^2+y^2}$

Real denominator

Let $z = x + iy$

$a \in \mathbb{R} \Rightarrow \dfrac{z}{a} = \dfrac{x}{a} + i\,\dfrac{y}{a}$

In general we can use complex conjugate for it:

Fact For any $z \in \mathbb{C}$

$z \cdot \bar{z}$ is a real number.

Proof Let $z = x + i \cdot y$ then $\bar{z} = x - iy$

then

$$z \cdot \bar{z} = (x + iy) \cdot (x - iy) =$$

$$= x^2 - (iy)^2 = x^2 - (\underbrace{-}_{= i^2} y^2) =$$

$$= x^2 + y^2 \in \mathbb{R}$$

$\blacksquare$

Back to fractions:

$$\frac{z_1}{z_2} = \frac{z_1 \cdot \overline{z_2}}{z_2 \cdot \overline{z_2}}$$

Usual complex multiplication

Real number

so division by it is also easy.

Absolute value:

Definition For a complex number $z = x + iy$ we define its absolute value (or ~~modulus~~ MODULUS) as

$$|z| = \sqrt{x^2 + y^2}$$

Important identity

$$\boxed{z \cdot \bar{z} = |z|^2}$$

$$\| \qquad\qquad \|$$

$$(x+iy)(x-iy) = x^2+y^2$$

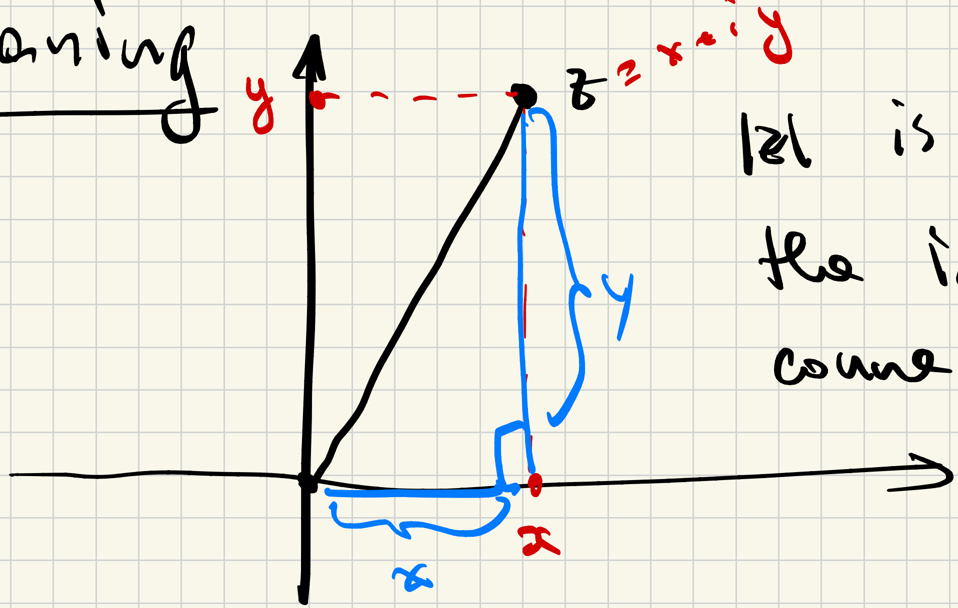
$$z = \frac{|z|^2}{\bar{z}} \Rightarrow \boxed{z^{-1} = \frac{\bar{z}}{|z|^2}} = \frac{x-iy}{x^2+y^2}$$

Remark. For purely real $z = x + i \cdot 0$

$$|x| = \sqrt{x^2 + 0^2} = \sqrt{x^2} = |x|.$$

- For any $z \in \mathbb{C}$ $|z| \in \mathbb{R}_{\geq 0}$

Geometric meaning



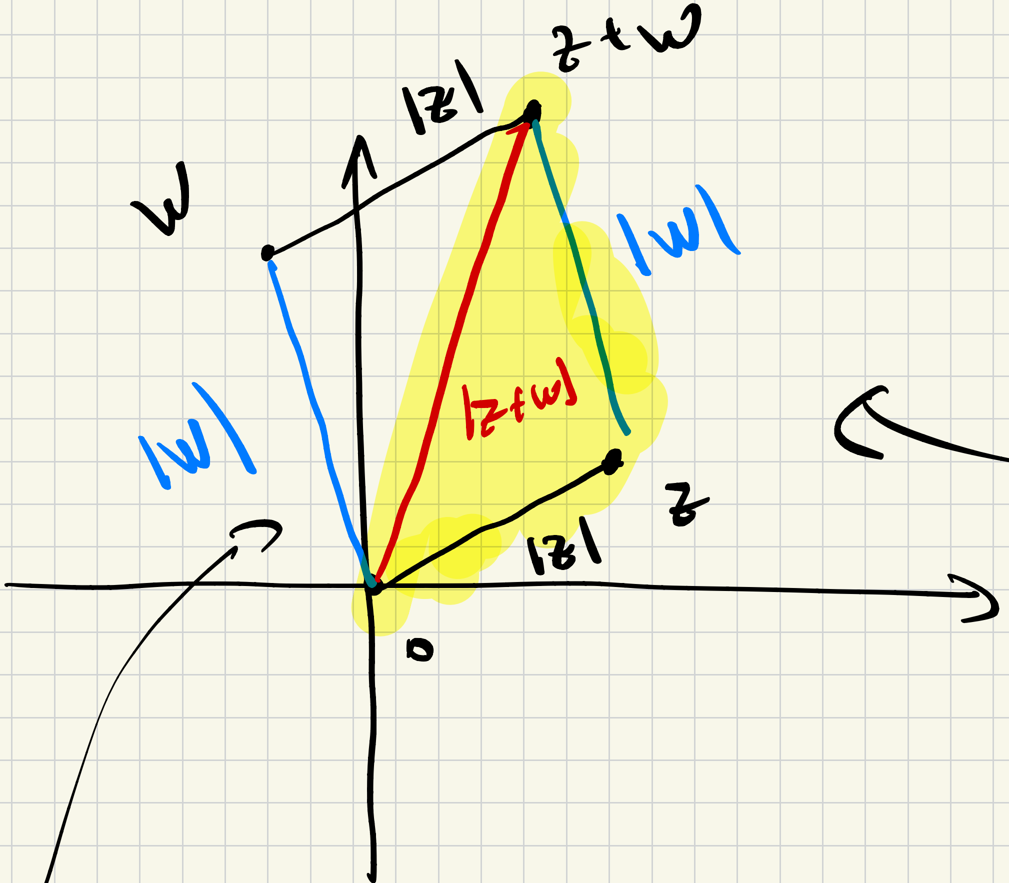
$|z|$ is length of the interval connecting z to 0.

by Pythagorus theorem:

$$|z|^2 = x^2 + y^2.$$

Triangle inequality

$$|z+w| \le |z| + |w|$$



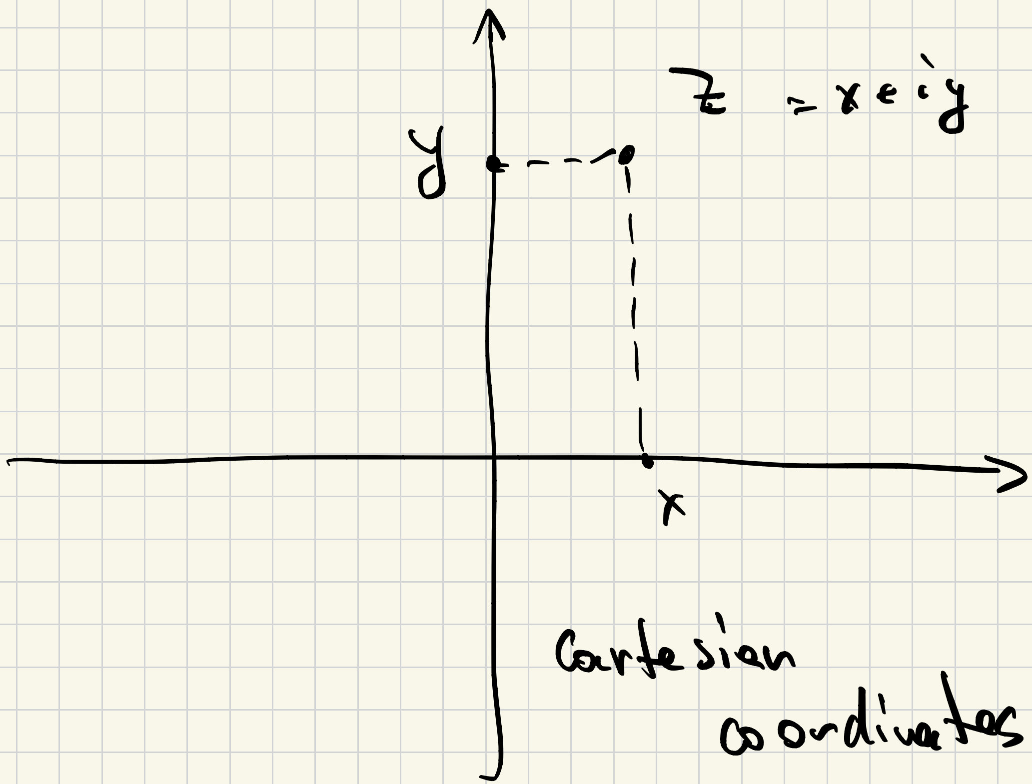
Parallelogram

triangle with side lengths

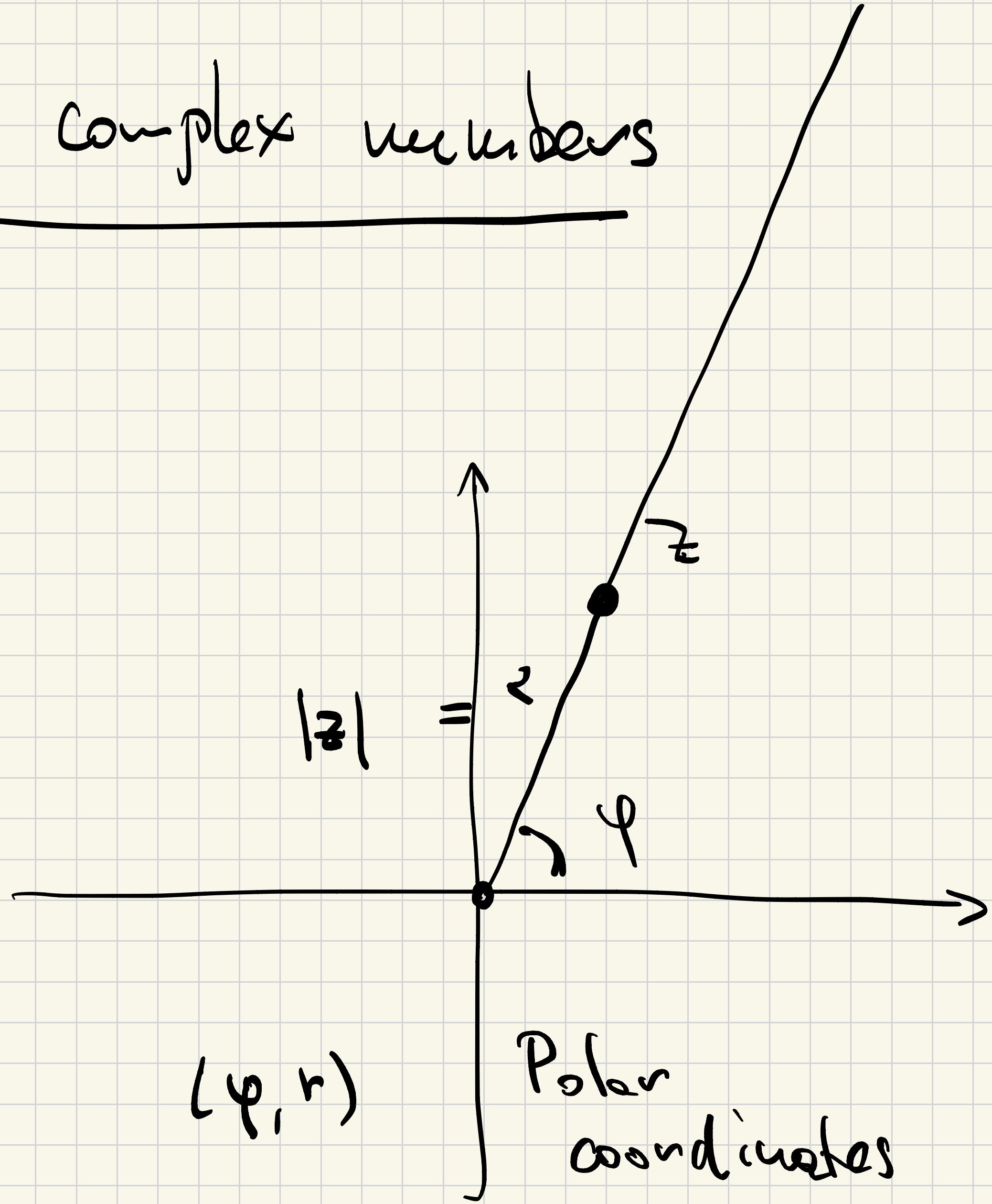
$|z+w|, \ |w|, \ |z|$

$\Rightarrow$ triangle inequality

Polar form of complex numbers



Cartesian coordinates



(φ, r)

Polar coordinates

Definition Argument of z is $\varphi \in \mathbb{R}$

s.t. $z = |z| \cdot (\cos\varphi + i\sin\varphi)$.

We denote by $\operatorname{Arg}(z)$.

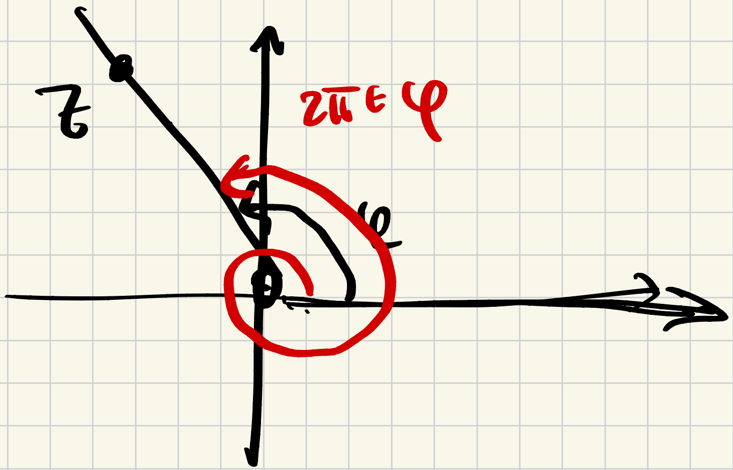
Remark: Not well defined.

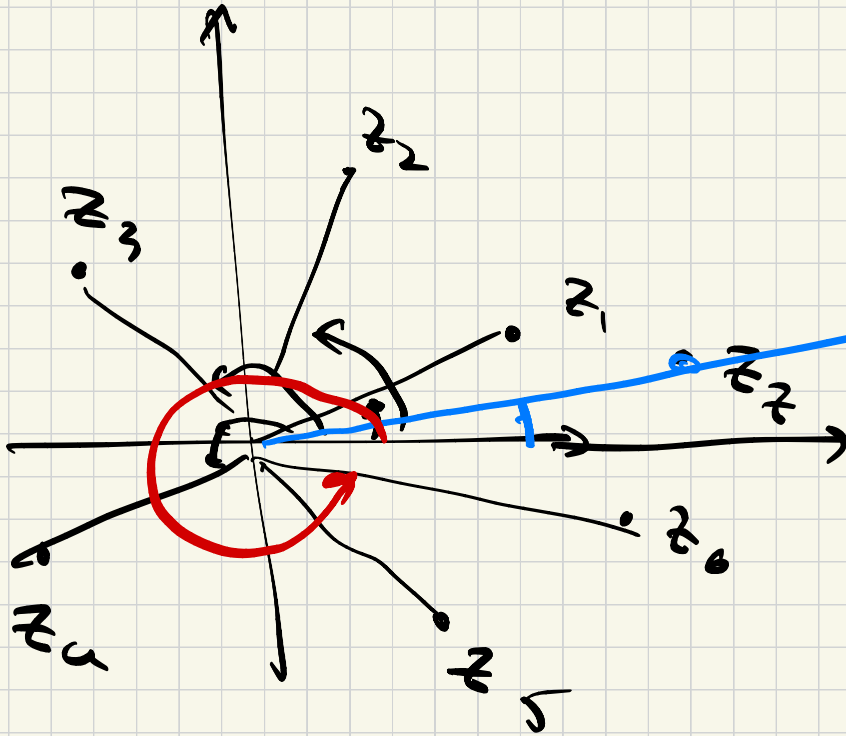
sin and cos are 2π - periodic functions.

so $|z| \cdot (\cos\varphi + i\sin\varphi) = |z|(\cos(2\pi+\varphi) + i\sin(2\pi+\varphi))$

If φ is an argument of z

the $\varphi + 2\pi$ is also an argument of z.

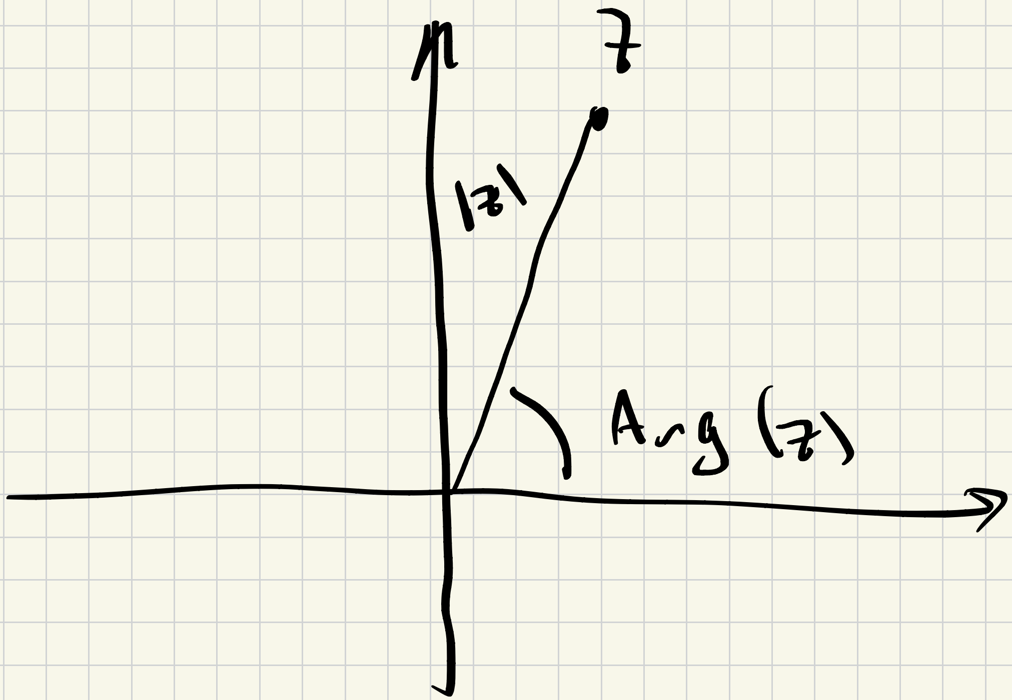




Polar presentation!

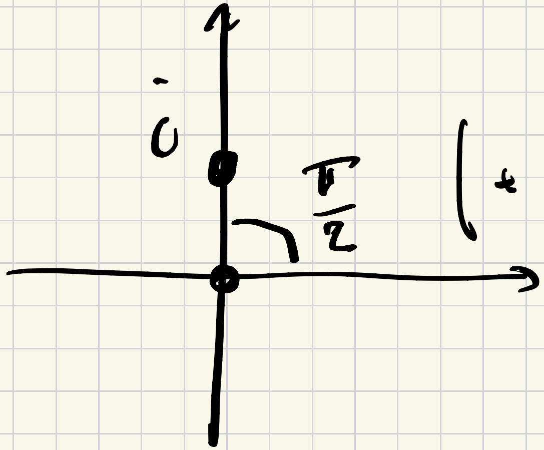
For any $z \in \mathbb{C}$ we have;

$$z = |z| \cdot \big(\cos(\operatorname{Arg} z) + i \sin(\operatorname{Arg} z) \big)$$



Examples

- i



$(+\,2\pi\cdot k)$

$k\in\mathbb{Z}$

$|i| = 1 = \sqrt{0^2+1^2} = 1$

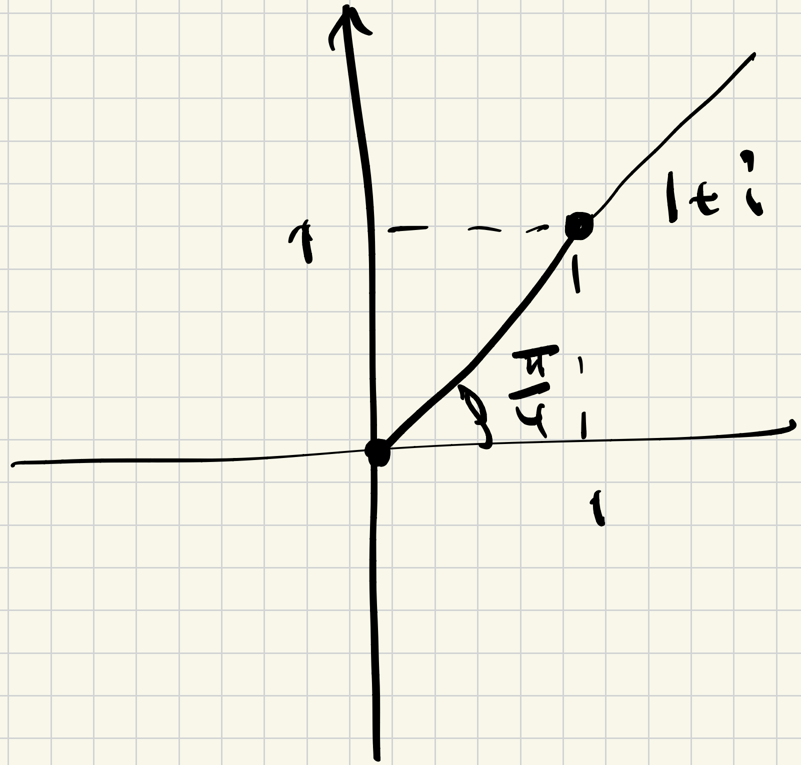
$\text{Arg}(i) = \frac{\pi}{2}$

Indeed $\quad i = 1\left(\underbrace{\cos\left(\frac{\pi}{2}\right)}_{=0} + i\,\underbrace{\sin\left(\frac{\pi}{2}\right)}_{=1}\right)$

- $1+i$

$$|1+i| = \sqrt{1^2+1^2} = \sqrt{2}$$

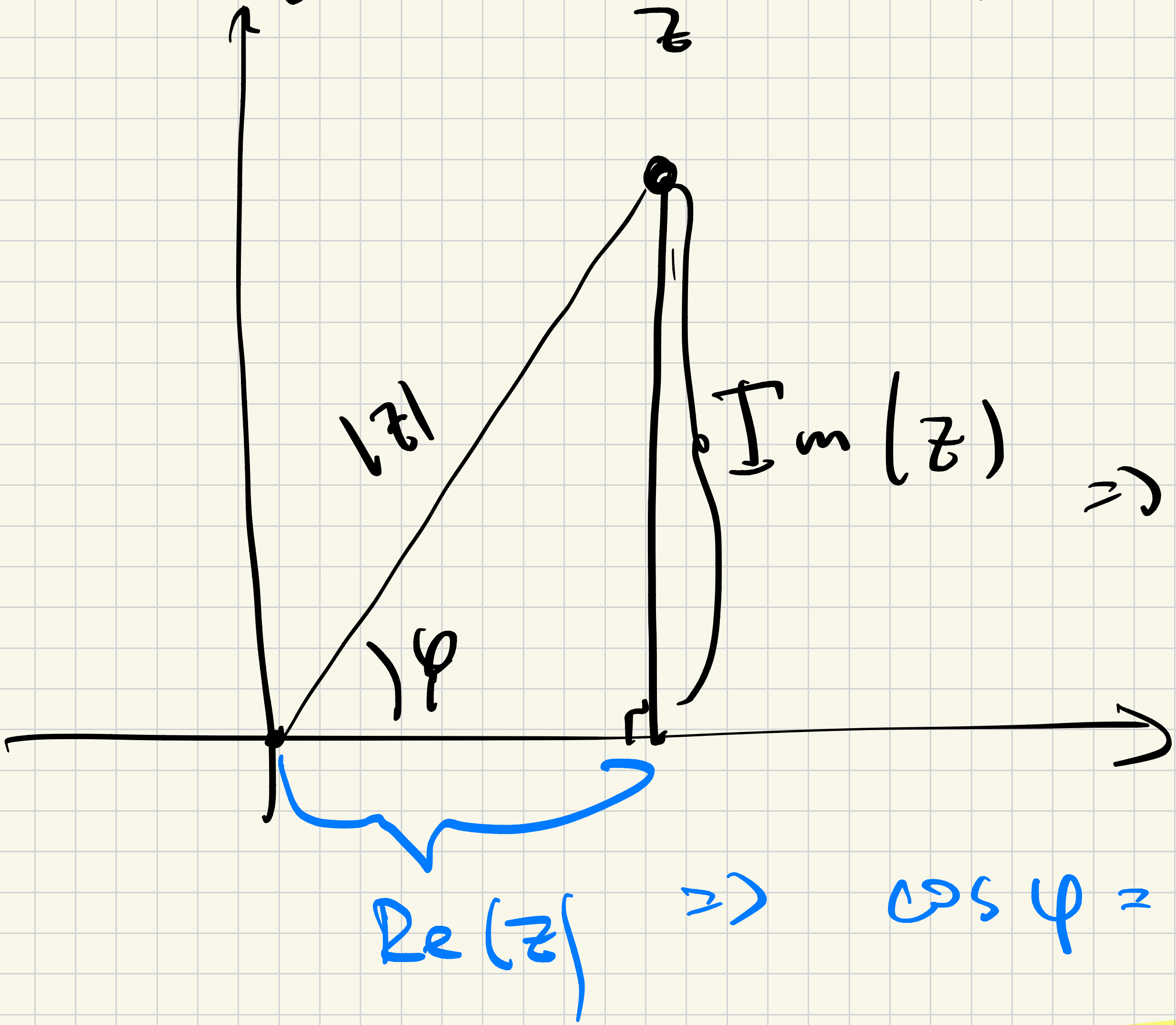
$$\operatorname{Arg}(1+i) = \frac{\pi}{4}$$



$$\Rightarrow \quad 1+i = \sqrt{2}\left(\underbrace{\cos\left(\frac{\pi}{4}\right)}_{=\frac{\sqrt{2}}{2}} + i\cdot \underbrace{\sin\left(\frac{\pi}{4}\right)}_{=\frac{\sqrt{2}}{2}}\right)$$

How to compute argument

If $z = x + iy$ with $x \geq 0$ $y \geq 0$



$$\sin\varphi = \frac{\operatorname{Im}(z)}{|z|}$$

$$\Rightarrow \varphi = \arcsin\left(\frac{\operatorname{Im}(z)}{|z|}\right)$$

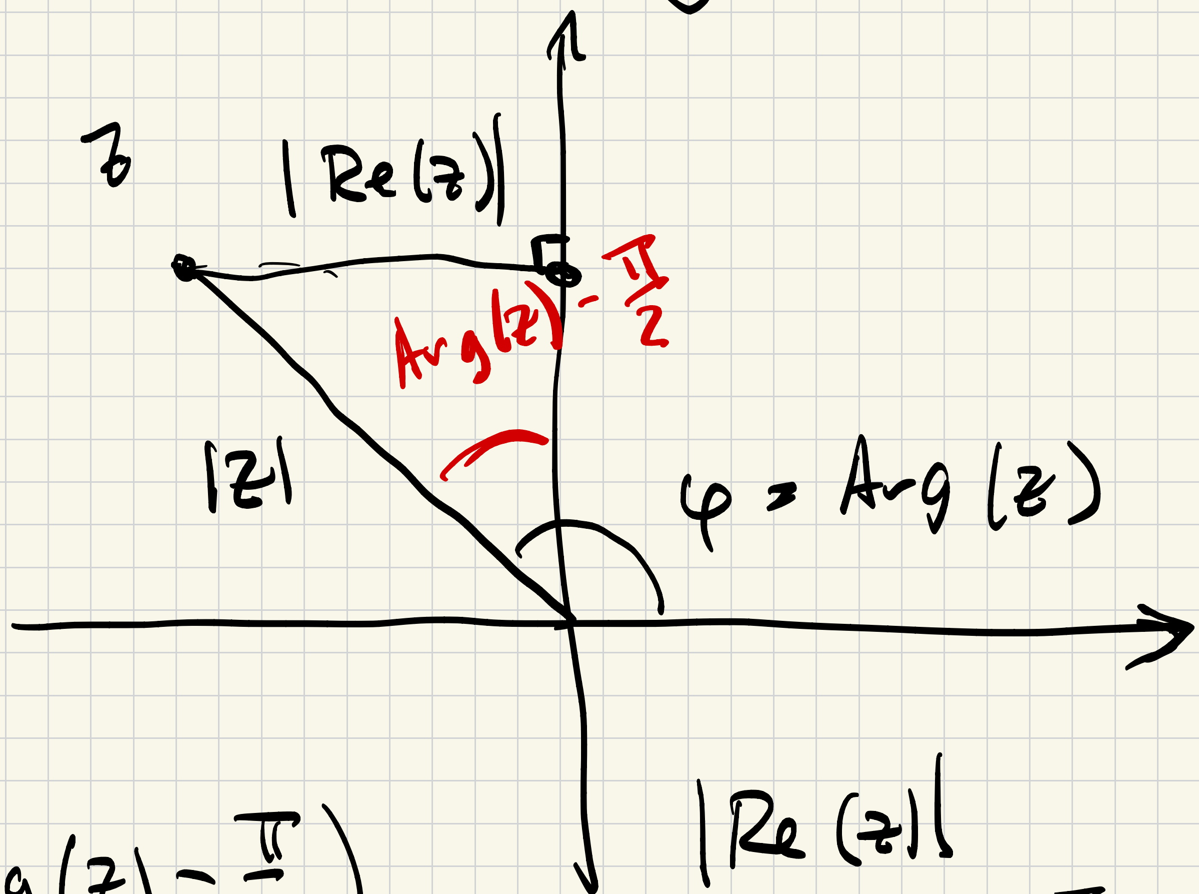
$$\Rightarrow \cos\varphi = \frac{\operatorname{Re}(z)}{|z|}$$

so $$\varphi = \arccos\left(\frac{\operatorname{Re}(z)}{|z|}\right)$$

In other situations you need to modify:

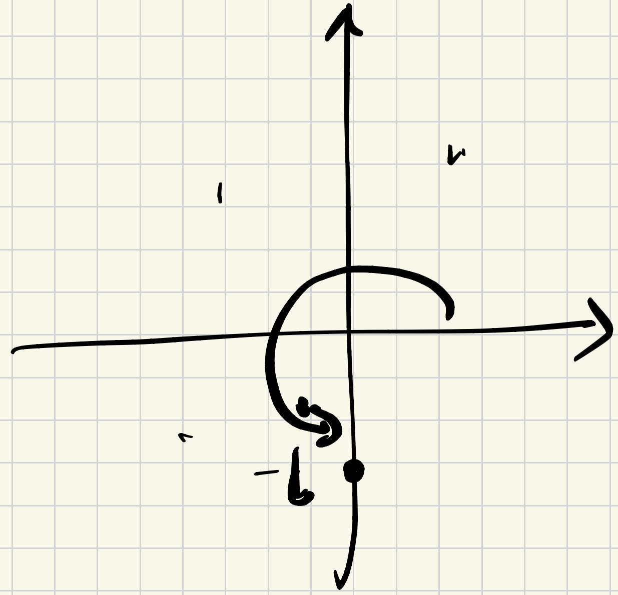
$Re(z) < 0$

$Im(z) > 0$



$$\sin\left(Arg(z) - \frac{\pi}{2}\right) = \frac{|Re(z)|}{|z|} = \frac{-Re(z)}{|z|}$$

$\bullet \quad -i$

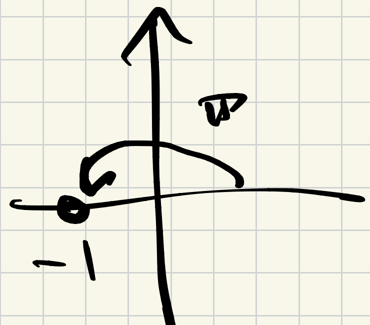


$$|-i| = 1$$

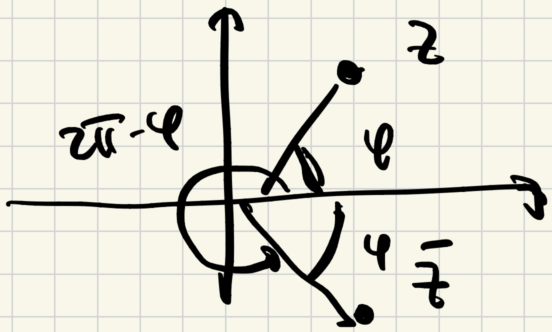
$$\operatorname{Arg}(-i) = \frac{3\pi}{2}$$

$$-i = 1 \cdot \left(\cos\left(\frac{3\pi}{2}\right) + i \sin\left(\frac{3\pi}{2}\right)\right)$$

$$-1 = 1 \cdot (\cos \pi + i \sin \pi)$$



Proposition



$$\operatorname{Arg} \bar{z} = -\operatorname{Arg}(z) \text{ or } 2\pi - \operatorname{Arg}(z)$$

$$\operatorname{Arg}(z_1 \cdot z_2) = \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2)$$

Proof: Computation using

$$\cos(\varphi + \psi) = \cos(\varphi) \cdot \cos(\psi) - \sin(\varphi)\sin(\psi)$$

$$\sin(\varphi + \psi) = \sin(\varphi) \cdot \cos(\psi) + \sin(\psi) \cdot \cos(\varphi)$$

Proposition

$$\mathrm{Arg}(z_1 \cdot z_2) = \mathrm{Arg}(z_1) + \mathrm{Arg}(z_2)$$

Proof: Computation using

$$\cos(\varphi + \psi) = \cos(\varphi) \cdot \cos(\psi) - \sin(\varphi)\sin(\psi)$$

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Exercise do the proof.

Product in the polar form

$$z_1 \cdot z_2 = |z_1| \cdot |z_2| \cdot$$

$$\cdot \left(\cos\left(\operatorname{Arg} z_1 + \operatorname{Arg} z_2\right) + i \sin\left(\operatorname{Arg} z_1 + \operatorname{Arg} z_2\right)\right)$$

<u>Example</u> Rotations of complex plane

If $|z| = 1 \Rightarrow z = \cos(\mathrm{Arg}(z)) + i\sin(\mathrm{Arg}(z))$

$(\cos^2\varphi + \sin^2\varphi = 1)$

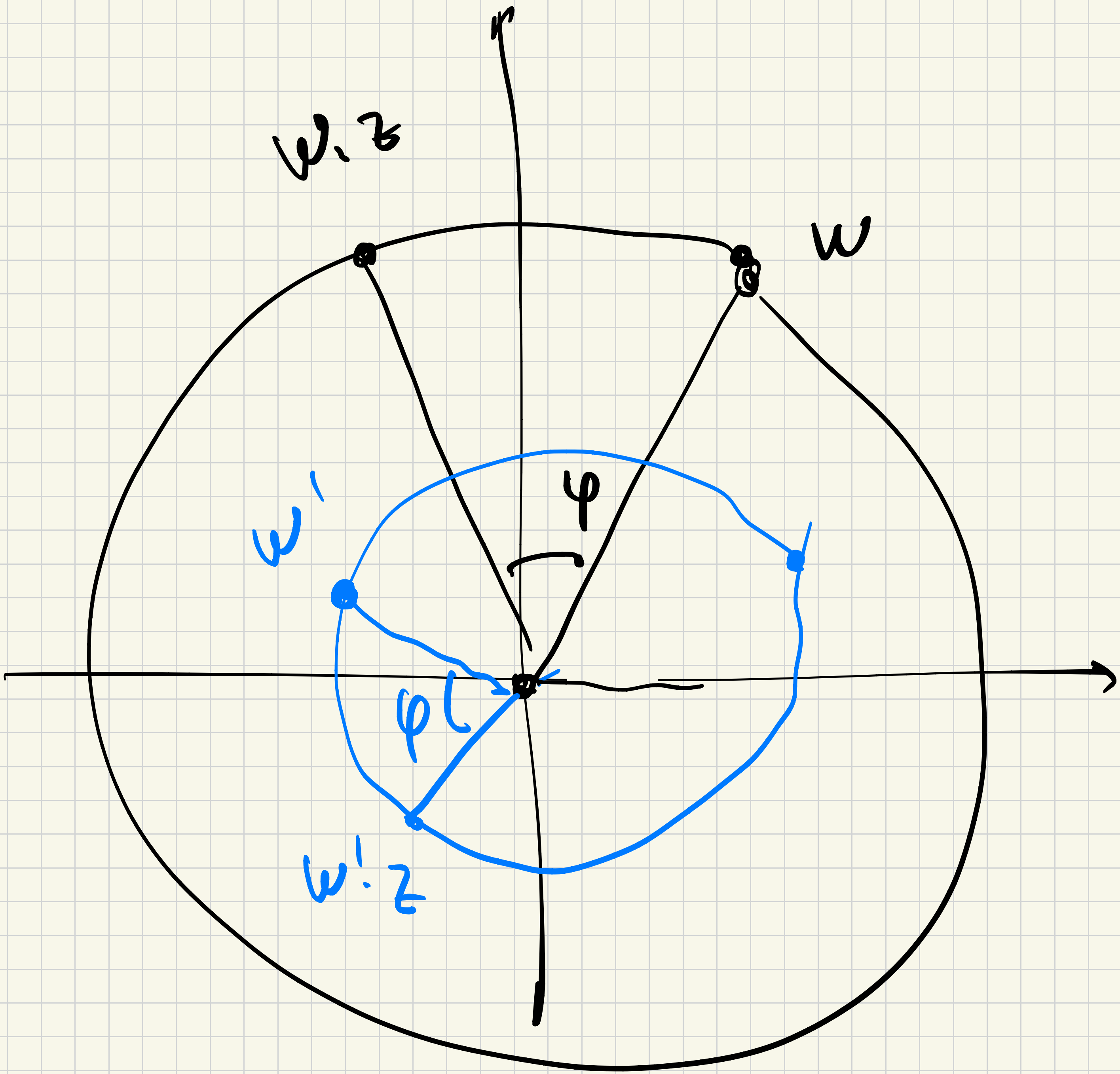
Multiplication by such $z = \cos\varphi + i\sin\varphi$ defines a map $\mathbb{C} \to \mathbb{C}$ given by $w \longmapsto z \cdot w$

Define a map: $\overset{\mathbb{R}^2}{\overset{\shortparallel}{\mathbb{C}}} \longrightarrow \overset{\mathbb{R}^2}{\overset{\shortparallel}{\mathbb{C}}}$
(or function)

$$w \longmapsto z \cdot w \quad \text{for}$$

$$z = \cos\varphi + i\sin\varphi$$

then this map is the rotation of $\mathbb{R}^2$ around 0 on angle φ.



w·z
w
φ
w'
φ
w'·z

Euler formula

Definition Complex exponential function

$$e^{z} = e^{(x+iy)} := e^{x}\cdot\left(\cos y + i\sin y\right)$$

In other words,

$$e^{z} = e^{\mathrm{Re}(z)}\cdot\left(\cos(\mathrm{Im}\, z) + i\sin(\mathrm{Im}\, z)\right)$$

Example

Euler's identity

$$\boxed{e^{\pi i} = -1}$$

$$e^{0+\pi\cdot i} = \underbrace{e^{0}}_{=1}\left(\underbrace{\cos\pi}_{=-1} + i\underbrace{\sin\pi}_{=0}\right)$$

Proposition

$$e^{z} = e^{\operatorname{Re}(z)} \cdot \left(\cos(\operatorname{Im} z) + i \sin(\operatorname{Im} z) \right)$$

- $|e^{z}| = e^{\operatorname{Re}(z)}$
- $\operatorname{Arg}(e^{z}) = \operatorname{Im}(z)$
- $e^{z + 2\pi i} = e^{z}$
- $\overline{e^{z}} = e^{\overline{z}}$
- $e^{z_1 + z_2} = e^{z_1} \cdot e^{z_2}$
- $(e^{z})^{n} = e^{n \cdot z}$
- $e^{z_1} = e^{z_2} \iff z_1 = z_2 + 2\pi i \cdot k \quad k \in \mathbb{Z}$